Derrick Stolee – Combinatorics Study Guide

Chapter 3: Subsets, Partitions, Permutations

The number of subsets is 2^n

The **binomial coefficient** $\binom{n}{k}$ is the number of subsets of size k from a set of n elements. Binomial coefficient identities:

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\quad \binom{n}{k} = \binom{n}{n-k}$ $\quad k\binom{n}{k} = n\binom{n-1}{k-1}$

$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ $\quad \sum_{k=0}^{n}\binom{n}{k} = 2^n$ $\quad \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

Thm (*Binomial Theorem*): $(1+t)^n = \sum_{k=0}^{n}\binom{n}{k}t^k$. *Strong form: $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Fact: For $n > 0$, the numbers of subsets of an n-set of even and of odd cardinality are equal. $\sum_{\substack{0\le k\le n\\ k \text{ odd}}}\binom{n}{k} = \sum_{\substack{0\le k\le n\\ k \text{ even}}}\binom{n}{k}$

Prop: If n is a multiple of 8, then the number of sets of size divisible by 4 is $2^{n-2} + 2^{(n-2)/2}$.

Thm (*Lucas' Theorem*): Let p be prime, and let $m = a_0 + a_1 p + \cdots + a_k p^k, n = b_0 + b_1 p + \cdots + b_k p^k$, where $0 \le a_i, b_i < p$ for $i = 0, \ldots, k-1$. Then $\binom{m}{n} \equiv \prod_{i=0}^{k}\binom{a_i}{b_i} \pmod{p}$.

Def: A **permutation** of a set X is a one-to-one mapping from X to itself.

Prop: The number of permutations of an n-set is $n!$

Prop: Any permutation can be written as the composition of cycles on pairwise disjoint subsets. the representation is unique, apart from the order of the factors, and the starting-points of the cycles.

Thm (*Stirling's Formula*): $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + O\left(\frac{1}{n}\right)\right)$.

Prop: $2^{2n}/(2n+1) \le \binom{2n}{n} \le 2^{2n}$

Lma: The number of n-tuples of non-negative integers $x_1, \ldots, x_n$ with $x_1 + \cdots + x_n = k$ is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

Lma: The number of choices of k objects from n with repetitions allowed and order not significant is equal to the number of ways of choosing n non-negative integers whose sum is k.

Thm: The number of selections of k objects from a set of n objects are given by the following table:

	Order Significant	Order Not Significant
Repetitions allowed	n^k	$\binom{n+k-1}{k}$
Repetitions not allowed	$n(n-1)\cdots(n-k+1) = \frac{n!}{k!}$	$\binom{n}{k}$

Prop: The number of ordered selections without repetition from a set of n objects is $\lfloor e \cdot n! \rfloor$, where e is the base of natural logarithms.

Def: A **relation** R on X is a subset of X^2. R is **reflexive** if $\forall x \in X, (x,x) \in R$. R is **irreflexive** if $\forall x \in X, (x,x) \notin R$. R is **symmetric** if $\forall x, y \in X, (x,y) \in R$ implies $(y,x) \in R$. R is **antisymmetric** if whenever $(x,y) \in R$ and $(y,x) \in R$, then $x = y$. R is **transitive** if, for all $x, y, z \in X, (x,y) \in R$ and $(y,z) \in R$ together imply $(x,z) \in R$.

Def: An **equivalence relation** is a reflexive, symmetric, and transitive relation. For $x \in X$ and an equivalence relation R, the **equivalence class** containing x is the set $R(x) = \{y \in X | (x,y) \in R\}$. A **partition** of X is a family of pairwise disjoint non-empty subsets whose union is X.

Thm: Let R be an equivalence relation on X. Then the equivalence classes of R form a partition of X. Conversely, given any partition of X, there is a unique equivalence relation on X whose equivalence classes are the parts of the partition.

Def: A relation R on X is a **partial order** if it is reflexive, antisymmetric, and transitive. A relation R is said to satisfy **trichotomy** if for any $x, y \in X$, one of the cases $(x,y) \in R, x = y$, or $(y,x) \in R$ holds. R is a **total order**, or **order**, if it is a partial order which satisfies trichotomy.

Def: A relation R is a **partial preorder**, or **pre-partial order**, if it is reflexive and transitive. A parial preorder satisfying trichotomy is a **preorder**.

[Skipping Section 3.9: Finite Topologies]

Thm (*Cayley's Theorem on Trees*): The number of labeled trees on n vertices is n^{n-2}.

Def: A **vertebrate** is a tree with two distinguished vertices called the **head** and **tail**. An **endofunction** on N is a function form N to itself.

Prop: The number of vertebrates and endofunctions on N are equal.

Def: The **Bell Number** B_n is the number of partitions of an n-set.

Thm (*Recurrence for Bell Numbers*): $B_0 = 1$. For $n \ge 1$, $B_n = \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}$.

Chapter 4: Recurrence relations and generating functions